ABSTRACT: In this paper we work with (co)homology of groups in the case \( R = \mathbb{Z}_2 \) (a field with two elements), which allows us to see these groups as vector spaces over \( \mathbb{Z}_2 \). We will study the invariant \( E(G, S, M) \), defined by Andrade & Fanti (1994), where \( G \) is a group, \( S = \{ S_i, i \in I \} \) is a non empty family of subgroups of \( G \) with \( [G : S_i] = \infty \) and \( M \) is a \( \mathbb{Z}_2G \)-module, in the case \( M = \mathbb{Z}_2 \). Some results are obtained when the group pair \((G, S)\) satisfies certain properties of duality and when \( G \) splits over a subgroup \( S \).

KEYWORDS: Cohomology of Groups, Duality, Splitting of Groups

1 Introduction

Let \( R \) be a commutative ring with identity element, \( G \) a group and \( M \) a \( RG \)-module, where \( RG \) denotes the group ring of \( G \) over \( R \). We will consider the concepts of homology \( H_*(G; M) \) and cohomology \( H^*(G; M) \) of \( G \) with coefficients in \( M \) as defined...
If $S = \{S_i, i \in I\}$ is a family of subgroups of $G$ we also consider the relative homology and relative cohomology groups $H_*(G, S; M)$ and $H^*(G, S; M)$, of the group pair $(G, S)$ with coefficients in $M$ (see Bieri & Eckmann (1978) or Andrade et al. (1999)). We will denote $\prod_{i \in I} H^k(S_i; M)$ by $H^k(S; M)$. The following result holds.

**Proposition 1.1** (Bieri & Eckmann, 1978) For a group pair $(G, S)$, $S = \{S_i, i \in I\}$ and a RG-module $M$ there is the following long exact sequence

$$0 \to H^0(G; M) \to H^0(S; M) \xrightarrow{\delta} H^1(G, S; M) \xrightarrow{\text{res}} H^1(G; M) \to \cdots$$

which is natural in the module $M$ and in the group pair $(G, S)$. \hfill \Box

In this paper, we will work with (co)homology of groups in the case $R = \mathbb{Z}_2$ (a field with two elements), which allows us to see these groups as vector spaces over $\mathbb{Z}_2$ and so we can use the concept of dimension of these spaces.

The purpose of this paper is to study the algebraic invariant $E(G, S, M)$ (due to Andrade & Fanti (1994)), where $G$ is a group, $S = \{S_i, i \in I\}$ is a non empty family of subgroups of $G$ with $[G : S_i] = \infty$ and $M$ is the trivial $\mathbb{Z}_2 G$-module $\mathbb{Z}_2$. In Andrade & Fanti (1994) the authors studied the invariant $E(G, S, \mathbb{Z}_2(G/S))$, denoted by $E(G, S)$, and the relations of this invariant with the ends $e(G), e(G, S)$, (Scott & Wall, 1979) and $\tilde{e}(G, S)$ (Kropholler & Roller, 1989). Here we will explore the relations between the invariant $E(G, S, \mathbb{Z}_2)$ with duality pairs and splittings of groups. For this, we recall some definitions and results.

**Definition 1.2** (Bieri, 1976) A group $G$ is called a duality group of dimension $n$ over $\mathbb{Z}_2$, or simply a $D^n$-group, if there exists a (left) $\mathbb{Z}_2 G$-module $C$ called the dualizing module of $G$, and natural isomorphisms

$$H^k(G; M) \cong H_{n-k}(G; C \otimes M)$$

for all integers $k$ and all $\mathbb{Z}_2 G$-modules $M$. In the special case where $C$ is isomorphic to $\mathbb{Z}_2$ as a $\mathbb{Z}_2 G$-module, we say that $G$ is a Poincaré duality group of dimension $n$ over $\mathbb{Z}_2$, or simply a $PD^n$-group.
**Definition 1.3** (Bieri & Eckmann,1978) A duality pair of dimension $n$, or simply a $D^n$-pair, consists of a group pair $(G, S)$ and a $\mathbb{Z}_2 G$-module $C$ where $S = \{S_i, i \in I\}$ is a finite family of $D^{n-1}$-subgroups of $G$ with dualizing module $C$ and natural isomorphisms

$$H^k(G; M) \simeq H_{n-k}(G, S; C \otimes M),$$
$$H^k(G, S; M) \simeq H_{n-k}(G; C \otimes M),$$

for all $\mathbb{Z}_2 G$-modules $M$ and all $k \in \mathbb{Z}$. $C$ is called the dualizing module of the $D^n$-pair $(G, S)$. If $C$ is isomorphic to $\mathbb{Z}_2$ as $\mathbb{Z}_2 G$-module, the duality pair $(G, S)$ is called a Poincaré duality pair, or simply a $PD^n$-pair.

**Remark 1.4** One can see in Bieri & Eckmann,(1978), that, if $(G, S)$ is a $D^n$-pair with dualizing module $C$ then $G$ is a $D^{n-1}$-group with dualizing module $\Delta \otimes C$ where $\Delta$ is the kernel of the map $\varepsilon : \mathbb{Z}_2(G/S) = \bigoplus_{i=1}^r \mathbb{Z}_2(G/S_i) \to \mathbb{Z}_2$ defined by $\varepsilon(gS_i) = 1$ and extended by linearity.

**Definition 1.5** (Bieri, 1976)) Let the groups $G_1$ and $G_2$ be given by presentations $G_k =\langle X_k; R_k \rangle$, $k = 1, 2$ where $X_k$ is a set of generators and $R_k$ a set of defining relations for $G_k$.

(a) If $T_1 \subset G_1$ and $T_2 \subset G_2$ are subgroups with a given isomorphism $\sigma : T_1 \simeq T_2$ then the free product $G_1 * T G_2$ of $G_1$ and $G_2$ with amalgamation subgroup $T = T_1 = T_2$ is given by

$$G_1 * T G_2 = \langle X_1, X_2; R_1, R_2, t = \sigma(t), \forall t \in T \rangle.$$

(b) If $G_1 =\langle X_1; R_1 \rangle$ and if $T, T'$ are subgroups of $G_1$ with a given isomorphism $\sigma : T \simeq T'$ then the HNN-group $G_1 *_{T, \sigma}$ over the base group $G_1$, with respect to $\sigma : T \simeq T'$ and with stable letter $p$, is given by

$$G_1 *_{T, \sigma} = \langle X_1, p; R_1, p^{-1} tp = \sigma(t), t \in T \rangle.$$

**Proposition 1.6** (Bieri, 1976)

(a) If $G = G_1 *_{T} G_2$ then one has the short exact sequence of $\mathbb{Z}_2 G$-modules

$$0 \to \mathbb{Z}_2(G/T) \to \mathbb{Z}_2(G/G_1) \oplus \mathbb{Z}_2(G/G_2) \to \mathbb{Z}_2 \to 0.$$
where $\alpha$ is given by $\alpha(xT) = (xG_1, xG_2)$, $x \in G$ and $\varepsilon$ is the augmentation.

(b) If $G = G_1 \ast_T \sigma$ then one has the short exact sequence of $\mathbb{Z}_2G$-modules

$$0 \rightarrow \mathbb{Z}_2(G/T) \xrightarrow{\alpha} \mathbb{Z}_2(G/G_1) \xrightarrow{\varepsilon} \mathbb{Z}_2 \rightarrow 0$$

where $\alpha$ is given by $\alpha(xT) = xG_1 - xpG_1$.

Definition 1.7 We say that a group $G$ splits over a subgroup $S$ if $G$ is either

(a) a non trivial product with amalgamation over $S$, i.e., $G = G_1 *_S G_2$ with $G_1 \neq S \neq G_2$ or
(b) an HNN-group with base group $S$, i.e., $G = G_1 *_{S, \sigma}$

Remark 1.8: It is true that, if $G = G_1 *_S G_2$ with $G_1 \neq S \neq G_2$ or $G = G_1 *_{S, \sigma}$ then $[G : G_i] = \infty$ (see Andrade & Fanti (2002)).

2 Definition of the invariant $E'(G, S)$

Definition 2.1 (Andrade & Fanti, 1994) Let $(G, S)$ be a group pair, $S = \{S_i, i \in I\}$ a family of subgroups of $G$ with $[G : S_i] = \infty, \forall i \in I$, and $M$ a $\mathbb{Z}_2G$-module. We define

$$E(G, S, M) = 1 + \dim \ker(res^G_S)$$

where $res^G_S : H^1(G; M) \xrightarrow{res^G_S} \prod_{i \in I} H^1(S_i; M)$ is the homomorphism that appears in the long exact sequence given by Proposition 1.1, which is induced in cohomology by the inclusions $S_i \rightarrow G$.

We have the following result, which generalizes Lemma 1.3 in Andrade & Fanti, (1994).

Lemma 2.2 Let $(G, S)$ be a group pair where $S = \{S_i; i \in I\}$ with $[G : S_i] = \infty, \forall i \in I$, and $M$ a $\mathbb{Z}_2G$-module. If the vectors spaces $H^0(G; M)$, $H^0(S; M) := \prod_{i \in I} H^0(S_i; M)$ and $H^1(G, S; M)$ have finite dimensions then

$$E(G, S, M) = 1 + \dim H^0(G; M) - \dim H^0(S; M) + \dim H^1(G, S; M).$$
Proof: The proof follows immediately from the long exact sequence of Proposition 1.1.

We will study $E(G, S, M)$ in the case $M$ is the trivial $\mathbb{Z}_2G$-module $\mathbb{Z}_2$. By simplicity, we will denote $E(G, S, \mathbb{Z}_2)$ by $E'(G, S)$. Thus, if $G$ is a group and $S = \{S_i, i \in I\}$ is a family of subgroups of $G$ with infinite index in $G$ we have:

$$E'(G, S) = 1 + \dim \ker \text{res}^G_S$$

where $\text{res}^G_S : H^1(G; \mathbb{Z}_2) \to \prod_{i \in I} H^1(S_i; \mathbb{Z}_2)$ is the restriction map.

Let us consider $S = \{S\}$ a family with only one subgroup. In this case we have the following result:

**Proposition 2.3** Let $G$ be a group and $S$ a subgroup of $G$ with $[G : S] = \infty$. Then $E'(G, S) = 1 + \dim H^1(G, S; \mathbb{Z}_2)$.

**Proof:** By Proposition 1.1, we have, for $M = \mathbb{Z}_2$ and $S = \{S\}$, the long exact sequence $0 \to H^0(G; \mathbb{Z}_2) \xrightarrow{\chi} H^0(S; \mathbb{Z}_2) \to H^1(G, S; \mathbb{Z}_2) \to H^1(G; \mathbb{Z}_2) \xrightarrow{l} \cdots$. It is easy to see that the map $\chi$ is an isomorphism and $l$ is the null map. Therefore $J$ is injective. Thus $H^1(G, S; \mathbb{Z}_2) \simeq \ker \text{res}^G_S$ and so, $E'(G, S) = 1 + \dim H^1(G, S; \mathbb{Z}_2)$.

It is interesting to note that several authors (see for example Andrade & Fanti, (1994), Kropholler & Roller, (1989) and Swarup, (1993)) have obtained, for "ends of pairs" $(G, S)$ where $G$ is a group and $S$ is a subgroup of $G$, formulas using $H^1(–)$ (absolute or relative cohomology).

### 3 The invariant $E'(G, S)$ and duality

In this section we study the invariant $E'(G, S)$ when $(G, S)$ is a duality pair. As we have seen in the last section, this invariant is defined when $[G : S] = \infty$, for all $S \in \mathcal{S}$. When we work with $D^n$-pairs this condition is not so restrictive as we will see in the next proposition. This result was proved for $PD^n$-pairs by Kropholler & Roller, (1988). Here we will prove it for $D^n$-pairs using other techniques.

Before that we will recall the concept of commensurability of subgroups. Let $S, T$ be subgroups of a group $G$. We say that $S$ is
commensurable with $T$ if $[S : S \cap T] < \infty$ and $[T : T \cap S] < \infty$. The set of all $g \in G$ such that $S$ is commensurable with $S^g = gSg^{-1}$ is denoted by $\text{Comm}_G(S)$.

**Theorem 3.1** Let $(G, S)$ be a $D^0$-pair with dualizing module $C$ and $S = \{S_i, i = 1, \ldots, r\}$. Then one of the following statements is true:

(i) $S$ consists of only one subgroup $S$ with $[G : S] = 2$ or $S$ consists of two copies of $G$.

(ii) $[G : S] = \infty$ and $\text{Comm}_G(S) = \{g \in G$ such that $S$ is commensurable with $S^g\} = S$ for all $S \in S$. Furthermore $S_j$ is not commensurable with $S^g_i$ for all $i \neq j$ and all $g \in G$.

**Proof:** First we suppose there is $S \in S$ such that $[G : S] < \infty$. Consider the $\mathbb{Z}_2G$-module $Ind_G^S \mathbb{Z}_2$ where $PS$ denotes the power set of $S$. We have that $PS$ may be viewed as $\text{Coind}_{\{1\}} G \mathbb{Z}_2$ and, since $[G : S] < \infty$, $Ind_G^S \mathbb{Z}_2 = \text{Coind}_{\{1\}} G \mathbb{Z}_2$ (Brown, 1982, III.5.9). Hence $Ind_G^S \mathbb{Z}_2 = \text{Coind}_{\{1\}} G \mathbb{Z}_2 = \text{Coind}_{\{1\}} G \mathbb{Z}_2 = PG$.

Consider the long exact sequence for the pair $(G, S)$ and the $\mathbb{Z}_2G$-module $PG$, $0 \rightarrow H^0(G; PG) \rightarrow H^0(S; PG) \rightarrow H^1(G, S; PG) \rightarrow H^1(G; PG) \rightarrow \cdots$. By the Shapiro’s Lemma,

$$H^k(G; PG) = \begin{cases} 0 & \text{se } k > 0 \\ \mathbb{Z}_2 & \text{se } k = 0 \end{cases}$$

and, using that $Ind_G^S \mathbb{Z}_2 = PG$, duality and Shapiro’s Lemma again, we have $H^1(G, S; PG) = H_{n-1}(G; C \otimes PG) = H_{n-1}(G; C \otimes Ind_G^S \mathbb{Z}_2) = H_{n-1}(G; \text{Coind}_{\{1\}} G \mathbb{Z}_2) = H_{n-1}(S; S \mathbb{Z}_2) = H^0(S; S \mathbb{Z}_2) \simeq \mathbb{Z}_2$. Thus we have the short exact sequence $0 \rightarrow \mathbb{Z}_2 \rightarrow H^0(S; PG) \rightarrow \mathbb{Z}_2 \rightarrow 0$ and therefore $H^0(S; PG) = \bigoplus_{i=1}^r H^0(S_i; PG) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Since $\mathbb{Z}_2 \simeq \{\emptyset, S_1\} \subset (PG)^{S_1} = H^0(S_1; PG)$, $S$ has at most two subgroups.

If $S = \{S_1, S_2\}$ then $(PG)^{S_1} \oplus (PG)^{S_2} = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and so $(PG)^{S_i} \simeq \mathbb{Z}_2$. Since $\emptyset, G \subset (PG)^{S_i}$ we have $S_1 = G$ for $i = 1, 2$ and so $S = \{G, G\}$.

If $S = \{S\}$ then $(PG)^S = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Hence $(PG)^S = \emptyset, G, S, S^G$. But, if $g_1 \in G - S$ then $S^G = Sg_1$ because $sSg_1 = Sg_1$, for all $s \in S$. Therefore $G = S \cup Sg_1$ and we have $[G : S] = 2$. 

Suppose now that \([G : S_i] = \infty\) for \(i = 1, \ldots, r\). We know that that

\[
\mathbb{Z}_2(G/S_i)^{S_j} = \begin{cases} 
0 & \text{se } i \neq j \\
\mathbb{Z}_2 & \text{se } i = j.
\end{cases}
\]

Now, let \(E\) be a transversal for the double cosets \(S_igS_i\) for \(i\) and \(j\) fixed and 1 denoting the coset \(S_jS_i\). By Mackey’s theorem (Brown, 1982, III.5.6(b)),

\[
\text{Res}_{S_j} \mathbb{Z}_2(G/S_i) = \bigoplus_{g \in E} \text{Ind}_{S_j \cap S_i}^{S_i} g \mathbb{Z}_2.
\]

Hence

\[
\mathbb{Z}_2(G/S_i)^{S_j} = H^0(S_j; \bigoplus_{g \in E} \text{Ind}_{S_j \cap S_i}^{S_i} g \mathbb{Z}_2)
= \bigoplus_{g \in E} H^0(S_j; \text{Ind}_{S_j \cap S_i}^{S_i} g \mathbb{Z}_2).
\]

When \(i = j\) and \(g = 1\), we have that \(H^0(S_i; \text{Ind}_{S_i}^{S_i} \mathbb{Z}_2) = \mathbb{Z}_2\) and so \(H^0(S_i; \text{Ind}_{S_i \cap S_i}^{S_i} g \mathbb{Z}_2) = 0\), for all \(g \neq 1, g \in E\). Therefore \([S_i : S_i \cap S_i] = \infty \forall g \in E, g \neq 1\), because, otherwise we would have, \(H^0(S_i; \text{Ind}_{S_i \cap S_i}^{S_i} g \mathbb{Z}_2) = H^0(S_i; \text{Coind}_{S_i \cap S_i}^{S_i} g \mathbb{Z}_2) = H^0(S_i \cap S_i^g; g \mathbb{Z}_2) \simeq \mathbb{Z}_2\) which is a contradiction. Hence \(S_i\) is not commensurable with \(S_i^g\), for all \(g \neq 1\) in \(E\). Since Mackey’s Theorem is independent of the transversal \(E\) considered, we have that \(S_i\) is not commensurable with \(S_i^g\), for all \(g \in G - S_i\) and therefore \(\text{Com}_G(S_i) = S_i\) for all \(i = 1, \ldots, r\).

If \(i \neq j\) then \(H^0(S_j; \text{Ind}_{S_j \cap S_i}^{S_i} g \mathbb{Z}_2) = 0\) for all \(g \in G\). Therefore, by the same argument above, we have \([S_j : S_j \cap S_i] = \infty, \forall g \in G \text{ and } S_j\) is not commensurable to any conjugate of \(S_i\) for \(i \neq j\).

\[\blacksquare\]

**Corollary 3.2** If \((G, S)\) is a \(D^n\)-pair with \([G : S] = \infty, \forall S \in S\), then \(S = N_G(S) = \text{Com}_G(S)\) for all \(S \in S\).

**Proof:** We have \(S \subseteq N_G(S) = \{g \in G : S^g = S\} \subseteq \text{Com}_G(S)\). So the result follows using (ii) of Theorem 3.1. \[\blacksquare\]
Proposition 3.3 If \((G, S)\) is a \(D^n\)-pair with dualizing module \(C\) and \([G : S] = \infty\), then

\[ E'(G, S) = 1 + \dim H_{n-1}(G; C). \]

Proof: Since \((G, S)\) is a \(D^n\)-pair we have that \(H^1(G, S; \mathbb{Z}_2) \simeq H_{n-1}(G; C) \otimes_{\mathbb{Z}_2} \mathbb{Z}_2 \simeq H_{n-1}(G; C)\). Thus, by Proposition 2.3, we have that \(E'(G, S) = 1 + \dim H_{n-1}(G; C)\). \(\blacksquare\)

Example 3.4 Let \(G = \langle a \rangle \ast \langle b \rangle \simeq \mathbb{Z} \ast \mathbb{Z}\) and \(S = \langle aba^{-1}b^{-1} \rangle \simeq \mathbb{Z}\). We have that \((G, S)\) is a \(PD^2\)-pair (Bieri & Eckmann, 1978) and then \(E'(G, S) = 1 + \dim H_1(G; \mathbb{Z}_2)\). But \(H_1(G; \mathbb{Z}_2) = \mathbb{Z}_2 \otimes \mathbb{Z}_2\). Hence we have \(E'(G, S) = 3\). It is interesting to note that, for this pair \((G, S)\), we have \(E(G, S) := E(G, S, \mathbb{Z}_2(G/S)) = 1\) (see Andrade & Fanti, (1994), example 2.11 (a)). Hence the invariants \(E'(G, S)\) and \(E(G, S)\) can provide different results.

Let \((G, S)\) be a \(PD^n\)-pair. The next theorem shows that, under these conditions, \(E'(G, S)\) is finite and there is a relation between this invariant and the number of elements of the family \(S\).

Theorem 3.5 Let \((G, S)\) be a \(PD^n\)-pair, \(S = \{S_i, i = 1, \ldots, r\}\), with \([G : S_i] = \infty\) and \(G\) finitely presented. Then \(E'(G, S) = 2 - r + \dim H_{n-1}(G; \mathbb{Z}_2) < \infty\).

Proof: If \((G, S)\) is a \(PD^n\)-pair, \(\{S_i, i = 1, \ldots, r\}\), we have that \(G\) is a \(D^{n-1}\)-group and consequently, by Bieri (1976), Theorem 9.2, \(G\) is of type \(FP\). Since \(G\) is finitely presented there is a finitely dominated \(K(G, 1)\)-complex (Brown, 1982, VIII.7.1). Thus \(H_*(G; \mathbb{Z}_2) = H_*(K(G, 1); \mathbb{Z}_2)\) are finitely generated. Now, we have that \(H^0(G, S; \mathbb{Z}_2) \simeq \mathbb{Z}_2\), \(H^i(S; \mathbb{Z}_2) = \bigoplus_{i=1}^{r} H^0(S_i; \mathbb{Z}_2) \simeq \bigoplus_{i=1}^{r} \mathbb{Z}_2\) and \(H^1(G, S; \mathbb{Z}_2) \simeq H_{n-1}(G; \mathbb{Z}_2)\). Since \(\dim H_{n-1}(G; \mathbb{Z}_2) < \infty\), we have, by Lemma 2.2, \(E'(G, S) = 2 - r + \dim H_{n-1}(G; \mathbb{Z}_2) < \infty\). \(\blacksquare\)

The next result follows immediately from the above theorem:

Corollary 3.6 If \((G, S), \{S_i, i = 1, \ldots, r\}\), is a \(PD^n\)-pair, with \(G\) finitely presented and \([G : S_i] = \infty\), \(i = 1, \ldots, r\), then \(r \leq 1 + \dim H_{n-1}(G; \mathbb{Z}_2) < \infty\). \(\blacksquare\)
Example 3.7 Let $G = \mathbb{Z} \ast \mathbb{Z}$. We have that $G$ is a $D^1$-group (Brown, 1982). Moreover, we have $H_1(G; \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Thus, if $S$ is a family of subgroups of $G$ with infinite index, then, in order for $(G, S)$ to be a $PD^2$-pair, the number of subgroups in the family $S$ has to be at most 3.

Remark 3.8 The last result is, to a certain extent, a generalization of the first part of the statement in Bieri & Eckmann, (1978), Proposition 11.1.

4 $E'(G, S)$ and splitting of groups

We will see now the relation between the invariant $E'(G, S)$ and splittings of groups. We have that, if $G = G_1 \ast_T G_2$ with $G_1 \neq T \neq G_2$ then $[G : G_1] = [G : G_2] = \infty$ and if $G = G_1 \ast_{T, \sigma} G_2$ then $[G : G_1] = \infty$. Thus, in this case, the invariant $E'(G, S)$ can be defined when $S$ is a family of subgroups of $G_1$ or $G_2$. In the following theorem we will give a necessary condition for $G$ to split over a subgroup $T$.

Theorem 4.1 Let $G$ be a group and $T$ a subgroup of $G$ and suppose that $G$ splits over $T$.

(a) If $G = G_1 \ast_T G_2$ then $E'(G, \{G_1, G_2\}) = 1$.

(b) If $G = G_1 \ast_{T, \sigma}$ then $E'(G, \{G_1\}) = 2$.

Proof: (a) It follows from the short exact sequence given by Proposition 1.6(a) that $\Delta = \ker \epsilon = \text{Im} \alpha \simeq \mathbb{Z}_2(G/T)$. Then, using Shapiro’s Lemma,

$$H^1(G, \{G_1, G_2\}; \mathbb{Z}_2) := H^0(G; \text{Hom}_{\mathbb{Z}_2}(\Delta, \mathbb{Z}_2)) \simeq H^0(G; \text{Hom}_{\mathbb{Z}_2}(\mathbb{Z}_2(G/T), \mathbb{Z}_2)) \simeq H^0(T; \mathbb{Z}_2) = \mathbb{Z}_2.$$ 

Thus, by Lemma 2.2, we have that $E'(G, \{G_1, G_2\}) = 1$.

(b) The proof is similar to (a), using Proposition 1.6.(b).

Corollary 4.2 Let $G$ be a group and $n$ an integer, $n > 1$, such that with dim $H_{n-1}(G; \mathbb{Z}_2) > 1$. Then:

(a) If $G_1$ and $G_2$ are subgroups of $G$ such that $(G, \{G_1, G_2\})$ is a $PD^n$-pair, then $G \neq G_1 \ast_T G_2$ for any subgroup $T$ of $G_1$ and $G_2$.

(b) If $G_1$ is a subgroup of $G$ such that $(G, \{G_1\})$ is a $PD^n$-pair, then $G \neq G_1 \ast_{T, \sigma}$, for any subgroup $T$ of $G_1$. 

Proof: If $(G, S)$ is a $PD^n$-pair then $H^1(G, S; \mathbb{Z}_2) \simeq H_{n-1}(G; \mathbb{Z}_2)$. Hence $E'(G, S) = 1 + \dim H_{n-1}(G; \mathbb{Z}_2) > 1$ if $S = \{G_1, G_2\}$, $E'(G, S) = 1 + \dim H_{n-1}(G; \mathbb{Z}_2) > 2$ if $S = \{G_1\}$ and the result follows by Theorem 4.1.

Example 4.3 Let $G = \langle s \rangle \ast \langle t \rangle \simeq \mathbb{Z} \ast \mathbb{Z}$, $S_1 = \langle s \rangle$, $S_2 = \langle t \rangle$ and $S = \langle sts^{-1}t^{-1} \rangle$. We have that $\dim H_1(G; \mathbb{Z}_2) = 2 > 1$. Now, since $(G, S)$ is a $PD^2$-pair, by Corollary 4.2, $G \neq S \ast T, \sigma$ for any subgroup $T$ of $S$. Moreover, since $G = S_1 \ast S_2$ we have that $(G, \{S_1, S_2\})$ is not a $PD^2$-pair, by the same corollary.

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RESUMO: Neste artigo trabalhamos com (co)homologia de grupos no caso $R = \mathbb{Z}_2$ (um corpo com dois elementos), o que nos permite ver tais grupos como espaços vetoriais sobre $\mathbb{Z}_2$. Nós estudaremos o invariante $E(G, S, M)$ (definido em Andrade & Fanti (1994)), onde $G$ é um grupo, $S = \{S_i, i \in I\}$ é uma família não vazia de subgrupos de $G$ com $[G : S_i] = \infty$ e $M$ é um $\mathbb{Z}_2 G$-módulo, quando $M = \mathbb{Z}_2$. Alguns resultados são obtidos quando o par grupo $(G, S)$ satisfaz certas propriedades de dualidade e quando $G$ se decompõe sobre um subgrupo $S$.

PALAVRAS-CHAVE: Cohomologia de Grupos, Dualidade, Decomposição de Grupos

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